



TITLE:

NONLINEAR MULTIPARAMETER EIGENVALUE PROBLEMS

AUTHOR(S):

Shibata, Tetsutaro

CITATION:

Shibata, Tetsutaro. NONLINEAR MULTIPARAMETER EIGENVALUE PROBLEMS. 数理解析研究所講究録 1996, 973: 40-52

ISSUE DATE:

1996-11

URL:

<http://hdl.handle.net/2433/60745>

RIGHT:

NONLINEAR MULTIPARAMETER EIGENVALUE PROBLEMS

Tetsutaro Shibata (柴田徹太郎)

The Division of Mathematical and Information Sciences

Faculty of Integrated Arts and Sciences

Hiroshima University, Higashi-Hiroshima, 739, Japan

1 Introduction

This paper is concerned with the nonlinear multiparameter problem

$$(1.1) \quad u''(r) + \frac{N-1}{r}u'(r) + \sum_{k=1}^n \mu_k f_k(u(r)) = \lambda g(u(r)), \quad 0 < r < 1,$$

$$(1.2) \quad u(r) > 0, \quad 0 \leq r < 1,$$

$$(1.3) \quad u'(0) = 0, \quad u(1) = 0,$$

where $N \geq 3$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in R_+^n$ ($n \geq 1$) and $\lambda \in R_+$ are parameters.

The aim of this paper is to establish asymptotic formulas of the variational eigenvalues $\lambda = \lambda(\mu, \alpha)$ obtained by Ljusternik-Schnirelman (LS) theory on general level set

$$N_{\mu, \alpha} := \{u \in X : \Lambda(\mu, u) := \frac{1}{2} \int_B |\nabla u|^2 dx - \sum_{k=1}^n \mu_k \int_B F_k(u(x)) dx = -\alpha \omega\},$$

where $X := W_0^{1,2}(B)$ is the usual real Sobolev space, $\alpha > 0$ is a parameter, ω is the measure of the unit sphere in R^N and $F_k(u) := \int_0^u f_k(s) ds$.

We assume the following conditions (A.1) - (A.3):

(A.1) f_k, g are locally Lipschitz continuous, odd in u and $f_k(u) \geq 0$, $g(u) > 0$ for $u > 0$.

(A.2) There exists a constant $0 < \epsilon_1$ such that for $1 \leq k \leq n$

$$(1.4) \quad f_k(u)u - (2 + \epsilon_1)F_k(u) \geq 0, \quad u \in R.$$

(A.3) There exist constants $1 < p_1 < p_2 < \dots < p_n < 1 + 4/N$ and $K_k > 0$ ($k = 0, 1, \dots, n$) such that as $u \rightarrow \infty$

$$(1.5) \quad \frac{g(u)}{u} \rightarrow K_0, \quad \frac{f_k(u)}{u^{p_k}} \rightarrow K_k.$$

Furthermore, there exist constants $1 < q_1 < q_2 < \dots < q_n < 1 + 4/N$ with $q_k \leq p_k$ and $J_k > 0$ ($k = 0, 1, \dots, n$) such that as $u \downarrow 0$

$$(1.6) \quad \frac{g(u)}{u} \rightarrow J_0, \quad \frac{f_k(u)}{u^{q_k}} \rightarrow J_k.$$

Our motivation comes from the problem of determining asymptotic direction of eigenvalues (the limit of the ratio of two eigenvalues) of linear multiparameter ordinary differential equations, which are sometimes studied by using Prüfer transformation. For linear theory, we refer to Faierman [3] and Turin [9] and the references therein.

Recently, in Shibata [7], the asymptotic direction for nonlinear two-parameter problems was studied for the simplest case of equation (1.1)-(1.3):

$$(1.7) \quad u''(x) + \mu u(x)^p = \lambda u(x)^q, \quad 0 < x < 1,$$

$$(1.8) \quad u(x) > 0, \quad 0 < x < 1,$$

$$(1.9) \quad u(0) = u(1) = 0.$$

Here $\mu, \lambda > 0$ are parameters and $1 \leq q < p < q + 2$ are constants. In [7], by LS-theory on general level set $N_{\mu, \alpha}$, the following asymptotic formula of λ as $\mu \rightarrow \infty$ for a fixed $\alpha > 0$ was given:

$$(1.10) \quad \lambda(\mu, \alpha) = C_1 \mu^{\frac{q+3}{p+3}} + o(\mu^{\frac{q+3}{p+3}}),$$

where

$$(1.11) \quad C_1 = \left\{ \left(\frac{q+1}{p+1} \right)^{\frac{q+3}{2(p-q)}} \frac{(p+3)(q+1)(p-q)\alpha}{2(2q-p+3)} \sqrt{\frac{2}{\pi(q+1)}} \frac{\Gamma\left(\frac{p+3}{2(p-q)}\right)}{\Gamma\left(\frac{q+3}{2(p-q)}\right)} \right\}^{\frac{2(p-q)}{p+3}}.$$

We shall extend this formula to our general situation.

2 Main Results

Let

$$(2.1) \quad \|u\|_X^2 := \frac{1}{\omega} \int_B |\nabla u|^2 dx, \quad \|u\|_p^p = \frac{1}{\omega} \int_B |u(x)|^p dx, \quad (u, v) := \frac{1}{\omega} \int_B u(x)v(x) dx,$$

$$(2.2) \quad \|u\|_\infty := \sup_{x \in B} |u(x)|, \quad \Phi_k(u) := \frac{1}{\omega} \int_B F_k(u(x)) dx, \quad \Psi(u) := \frac{1}{\omega} \int_B G(u(x)) dx.$$

For a given $(\mu, \alpha) \in R_+^{n+1}$, $\lambda = \lambda(\mu, \alpha)$ is called the variational eigenvalue if the following conditions (B.1) - (B.2) are satisfied:

$$(B.1) \quad (\mu, \alpha, \lambda(\mu, \alpha), u_{\mu, \alpha}(x)) \in R_+^{n+2} \times N_{\mu, \alpha} \text{ satisfies (1.1) - (1.3).}$$

$$(B.2)$$

$$(2.3) \quad \Psi(u_{\mu, \alpha}) = \beta(\mu, \alpha) := \inf_{u \in N_{\mu, \alpha}} \Psi(u).$$

$\lambda(\mu, \alpha)$ is explicitly represented as

$$(2.4) \quad \lambda(\mu, \alpha) = \frac{2\alpha + \sum_{k=1}^n \mu_k ((f_k(u_{\mu, \alpha}), u_{\mu, \alpha}) - 2\Phi_k(u_{\mu, \alpha}))}{(g(u_{\mu, \alpha}), u_{\mu, \alpha})}.$$

Indeed, multiplying (1.1) by $u_{\mu,\alpha}$ and integration by parts, we obtain

$$(2.5) \quad -\|u_{\mu,\alpha}\|_X^2 + \sum_{k=1}^n \mu_k (f_k(u_{\mu,\alpha}), u_{\mu,\alpha}) = \lambda(\mu, \alpha) (g(u_{\mu,\alpha}), u_{\mu,\alpha}).$$

This along with the fact that $u_{\mu,\alpha} \in N_{\mu,\alpha}$ yields (2.4).

Now we introduce (C-i) and (D-i)-condition for a sequence $\{(\mu, \alpha)\} \subset R_+^{n+1}$:

(C-i) Let $1 \leq i \leq n$ be fixed. Then

$$(2.6) \quad \alpha \mu_i^{\frac{2}{p_i-1}} \rightarrow \infty,$$

$$(2.7) \quad \alpha^2 \mu_i^{N-2} \rightarrow \infty.$$

Furthermore, for $k \neq i$

$$(2.8) \quad \mu_k \alpha^{\frac{2(p_k-p_i)}{N+2-p_i(N-2)}} \mu_i^{-\frac{N+2-p_k(N-2)}{N+2-p_i(N-2)}} \rightarrow 0.$$

(D-i) Let $1 \leq i \leq n$ be fixed. Then

$$(2.9) \quad \alpha \mu_i^{\frac{2}{q_i-1}} \rightarrow \infty,$$

$$(2.10) \quad \alpha^2 \mu_i^{N-2} \rightarrow 0.$$

Furthermore, for $k \neq i$

$$(2.11) \quad \mu_k \alpha^{\frac{2(q_k-q_i)}{N+2-q_i(N-2)}} \mu_i^{-\frac{N+2-q_k(N-2)}{N+2-q_i(N-2)}} \rightarrow 0.$$

Finally, let w denote the ground state solution of the following nonlinear scalar field equation, which uniquely exists:

$$(2.12) \quad -\Delta w = w^{p_i} - w \quad \text{in } R^N,$$

$$(2.13) \quad w > 0 \quad \text{in } R^N,$$

$$(2.14) \quad \lim_{|x| \rightarrow \infty} w(x) = 0.$$

Moreover, let W be the ground state of (2.12)-(2.14) with p_i replaced by q_i .

Now we state our results.

Theorem 2.1 Assume (A.1) - (A.3). Then the following asymptotic formula holds for $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfying (C-i):

$$(2.15) \quad \lambda(\mu, \alpha) = C_2 (\alpha \mu_i^{\frac{2}{p_i-1}})^{\frac{2(p_i-1)}{N+2-(N-2)p_i}} + o\left((\alpha \mu_i^{\frac{2}{p_i-1}})^{\frac{2(p_i-1)}{N+2-(N-2)p_i}}\right)$$

where

$$(2.16) \quad C_2 = K_0^{-1} K_i^{\frac{4}{N+2-(N-2)p_i}} \left\{ \frac{N+2-(N-2)p_i}{(4+N-Np_i)\|w\|_{L^2(R^N)}^2} \right\}^{\frac{2(p_i-1)}{N+2-(N-2)p_i}}.$$

Theorem 2.2 Assume (A.1) - (A.3). Then the following asymptotic formula holds for $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfying (D-i):

$$(2.17) \quad \lambda(\mu, \alpha) = C_3(\alpha\mu_i^{\frac{2}{q_i-1}})^{\frac{2(q_i-1)}{N+2-(N-2)q_i}} + o\left((\alpha\mu_i^{\frac{2}{q_i-1}})^{\frac{2(q_i-1)}{N+2-(N-2)q_i}}\right)$$

where

$$(2.18) \quad C_3 = J_0^{-1} J_i^{\frac{4}{N+2-(N-2)q_i}} \left\{ \frac{N+2-(N-2)q_i}{(4+N-Nq_i)\|W\|_{L^2(R^N)}^2} \right\}^{\frac{2(q_i-1)}{N+2-(N-2)q_i}}.$$

Typical examples of f_k, g are as follows.

Example 2.3 Let $1 < p_1 < \dots < p_i < \dots < p_n < 1 + 4/N$ and $1 < q_1 < \dots < q_i < \dots < q_n < 1 + 4/N$ with $q_k \leq p_k$. Then

$$(2.19) \quad f_k(u) = |u|^{p_k-1}u, \quad g(u) = u,$$

$$(2.20) \quad f_k(u) = (1 + |u|^{p_k-q_k})|u|^{q_k-1}u, \quad g(u) = u.$$

satisfies (A.1) - (A.3).

3 Fundamental lemmas for Theorem 2.1

Since Theorem 2.2 can be obtained by the similar arguments as those used to prove Theorem 2.1, we shall give a proof of Theorem 2.1. The existence of variational eigenvalues can be proved by using the result of Zeidler [10]. We begin with the basic equality which will play important roles. Let $\sigma_{\mu,\alpha} = \max_{0 \leq r \leq 1} u_{\mu,\alpha}(r)$ ($= u_{\mu,\alpha}(0)$).

Lemma 3.1 The following equality holds for $0 \leq r \leq 1$:

$$(3.1) \quad \begin{aligned} \frac{1}{2}(u'_{\mu,\alpha}(r))^2 + \int_0^r \frac{N-1}{s} u'_{\mu,\alpha}(s)^2 ds + R(\mu, \alpha, u_{\mu,\alpha}(r)) \\ = R(\mu, \alpha, \sigma_{\mu,\alpha}) = \frac{1}{2}(u'_{\mu,\alpha}(1))^2 + \int_0^1 \frac{N-1}{s} u'_{\mu,\alpha}(s)^2 ds > 0, \end{aligned}$$

where

$$(3.2) \quad R(\mu, \alpha, u) = \sum_{k=1}^n \mu_k F_k(u) - \lambda(\mu, \alpha) G(u).$$

This lemma can be obtained by direct calculation. Hence, we omit the proof. Let

$$(3.3) \quad h_0(u) := g(u) - u, \quad h_k(u) := f_k(u) - u^{p_k},$$

$$(3.4) \quad H_0(u) := \int_0^u h_0(s) ds, \quad H_k(u) := \int_0^u h_k(s) ds.$$

Then by (1.5) we see that as $u \rightarrow \infty$

$$(3.5) \quad \frac{h_0(u)}{u}, \quad \frac{h_k(u)}{u^{p_k}}, \quad \frac{H_0(u)}{u^2}, \quad \frac{H_k(u)}{u^{p_k+1}} \rightarrow 0.$$

Lemma 3.2 Assume that $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then

$$(3.6) \quad \lambda(\mu, \alpha) \geq C(\alpha\mu_i^{\frac{2}{p_i-1}})^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}.$$

Proof. Let $\tau = \tau_{\mu, \alpha} := (\alpha\mu_i^{\frac{2}{p_i-1}})^{\frac{(p_i-1)}{N+2-p_i(N-2)}}$. Furthermore, let $w_\tau(S)$ satisfy

$$(3.7) \quad w_\tau''(S) + \frac{N-1}{S}w_\tau'(S) + w_\tau(S)^{p_i} - w_\tau(S) = 0, \quad 0 < S < \tau_{\mu, \alpha},$$

$$(3.8) \quad w_\tau(S) > 0, \quad 0 \leq S < \tau_{\mu, \alpha},$$

$$(3.9) \quad w_\tau'(0) = w_\tau(\tau_{\mu, \alpha}) = 0.$$

We write, for simplicity,

$$\|w_\tau\|_X^2 := \int_0^{\tau_{\mu, \alpha}} S^{N-1} w_\tau'(S)^2 dS, \quad \|w_\tau\|_p^p := \int_0^{\tau_{\mu, \alpha}} S^{N-1} |w_\tau(S)|^p dS.$$

We know from Kwong [6] that there uniquely exists a solution w_τ , and since $\tau_{\mu, \alpha} \rightarrow \infty$ by (2.6), $\|w_\tau\|_2, \|w_\tau\|_\infty \leq C$ for $\{(\mu, \alpha)\}$. Put

$$(3.10) \quad d_{\mu, \alpha} := (\alpha\mu_i^{\frac{N-2}{2}})^{\frac{2}{N+2-p_i(N-2)}}, \quad U_{\mu, \alpha}(s) := c_{\mu, \alpha} d_{\mu, \alpha} w_\tau(S), \quad s := \tau_{\mu, \alpha}^{-1} S,$$

where $c_{\mu, \alpha} := \inf\{t > 0 : td_{\mu, \alpha} w_\tau(\tau_{\mu, \alpha} s) \in N_{\mu, \alpha}\}$. Then $\Xi(t) := \Lambda(\mu, td_{\mu, \alpha} w_\tau(\tau_{\mu, \alpha} s)) \rightarrow \infty$ as $t \rightarrow \infty$ and $\Xi(0) = 0$, we see that $c_{\mu, \alpha}$ exists. We first show that

$$(3.11) \quad C^{-1} \leq c_{\mu, \alpha} \leq C.$$

By (2.8), we obtain for $k \neq i$

$$(3.12) \quad \alpha^{-1} \mu_k d_{\mu, \alpha}^{p_k+1} \tau_{\mu, \alpha}^{-N} = \mu_k \alpha^{\frac{2(p_k-p_i)}{N+2-p_i(N-2)}} \mu_i^{-\frac{N+2-p_k(N-2)}{N+2-p_i(N-2)}} \rightarrow 0,$$

$$(3.13) \quad d_{\mu, \alpha}^2 \tau_{\mu, \alpha}^{2-N} = \mu_i d_{\mu, \alpha}^{p_i+1} \tau_{\mu, \alpha}^{-N} = \alpha.$$

We have by (3.12) and (3.13)

$$\begin{aligned} (3.14) \quad -\alpha &= \Lambda(\mu, U_{\mu, \alpha}) = \frac{1}{2} c_{\mu, \alpha}^2 d_{\mu, \alpha}^2 \tau_{\mu, \alpha}^{2-N} \|w_\tau\|_X^2 - \frac{1}{p_i+1} \mu_i c_{\mu, \alpha}^{p_i+1} d_{\mu, \alpha}^{p_i+1} \tau_{\mu, \alpha}^{-N} \|w_\tau\|_{p_i+1}^{p_i+1} \\ &\quad - \sum_{k \neq i}^n \frac{1}{p_k+1} \mu_k c_{\mu, \alpha}^{p_k+1} d_{\mu, \alpha}^{p_k+1} \tau_{\mu, \alpha}^{-N} \|w_\tau\|_{p_k+1}^{p_k+1} - \sum_{k=1}^n \mu_k \int_0^1 s^{N-1} H_k(U_{\mu, \alpha}(s)) ds \\ &= \frac{1}{2} c_{\mu, \alpha}^2 \alpha \|w_\tau\|_X^2 - \frac{1}{p_i+1} c_{\mu, \alpha}^{p_i+1} \alpha \|w_\tau\|_{p_i+1}^{p_i+1} \\ &\quad - \sum_{k \neq i}^n \frac{o(1)}{p_k+1} c_{\mu, \alpha}^{p_k+1} \alpha \|w_\tau\|_{p_k+1}^{p_k+1} - \sum_{k=1}^n \mu_k \int_0^1 s^{N-1} H_k(U_{\mu, \alpha}(s)) ds. \end{aligned}$$

At first, assume that $c_{\mu,\alpha} \rightarrow \infty$. Since $c_{\mu,\alpha}d_{\mu,\alpha} \rightarrow \infty$, we obtain by (2.8) and (3.12) that

$$\begin{aligned}
 (3.15) \quad & \mu_k \int_0^1 s^{N-1} H_k(U_{\mu,\alpha}(s)) ds \\
 &= \mu_k \tau_{\mu,\alpha}^{-N} \int_0^{\tau_{\mu,\alpha}} S^{N-1} \frac{H(c_{\mu,\alpha} d_{\mu,\alpha} w_\tau(S))}{(c_{\mu,\alpha} d_{\mu,\alpha} w_\tau(S))^{p_k+1}} (c_{\mu,\alpha} d_{\mu,\alpha} w_\tau(S))^{p_k+1} dS \\
 &= o(1) \mu_k c_{\mu,\alpha}^{p_k+1} d_{\mu,\alpha}^{p_k+1} \tau_{\mu,\alpha}^{-N} \|w_\tau\|_{p_k+1}^{p_k+1}.
 \end{aligned}$$

Then we obtain by (3.14) and (3.15) that

$$\begin{aligned}
 (3.16) \quad & -\alpha = \Lambda(\mu, U_{\mu,\alpha}) \\
 &= \alpha c_{\mu,\alpha}^2 \left\{ \frac{1}{2} \|w_\tau\|_X^2 - \frac{1+o(1)}{p_i+1} c_{\mu,\alpha}^{p_i-1} \|w_\tau\|_{p_i+1}^{p_i+1} - o(1) \sum_{k \neq i}^n \frac{1+o(1)}{p_k+1} c_{\mu,\alpha}^{p_k-1} \|w_\tau\|_{p_k+1}^{p_k+1} \right\}.
 \end{aligned}$$

Since $c_{\mu,\alpha} \rightarrow \infty$, this is a contradiction. Thus, $c_{\mu,\alpha} \leq C$.

Next, assume that $c_{\mu,\alpha} \rightarrow 0$. Then there are two cases to consider.

Case 1: If $c_{\mu,\alpha}d_{\mu,\alpha} \rightarrow \infty$, then by the same argument as that used above, we also obtain (3.16). Since $c_{\mu,\alpha} \rightarrow 0$, this is a contradiction.

Case 2: Assume that $c_{\mu,\alpha}d_{\mu,\alpha} \leq C$. By (1.6) we have for $u \leq C$

$$(3.17) \quad f_k(u) \leq C u^{q_k}, \quad F_k(u) \leq C u^{q_k+1}.$$

Then since $q_k \leq p_k$ and $d_{\mu,\alpha} \rightarrow \infty$, we have by (3.12) and (3.13) that for $1 \leq k \leq n$

$$\begin{aligned}
 (3.18) \quad \Phi_k(U_{\mu,\alpha}) &= \mu_k \int_0^1 s^{N-1} F_k(c_{\mu,\alpha} d_{\mu,\alpha} w_\tau(\tau_{\mu,\alpha} s)) ds \leq C \mu_k c_{\mu,\alpha}^{q_k+1} d_{\mu,\alpha}^{q_k+1} \tau_{\mu,\alpha}^{-N} \|w_\tau\|_{q_k+1}^{q_k+1} \\
 &\leq C c_{\mu,\alpha}^{q_k+1} (\mu_k d_{\mu,\alpha}^{p_k+1} \tau_{\mu,\alpha}^{-N} \alpha^{-1}) \alpha = C c_{\mu,\alpha}^{q_k+1} \alpha.
 \end{aligned}$$

Then

$$(3.19) \quad -\alpha = \Lambda(\mu, U_{\mu,\alpha}) \geq \alpha c_{\mu,\alpha}^2 \left\{ \frac{1}{2} \|w_\tau\|_X^2 - C \sum_{k=1}^n c_{\mu,\alpha}^{q_k-1} \right\}.$$

Since $c_{\mu,\alpha} \rightarrow 0$, this is a contradiction. Hence, we obtain (3.11). Now we obtain

$$\begin{aligned}
 (3.20) \quad & \frac{2\alpha}{\lambda(\mu, \alpha)} \leq (g(u_{\mu,\alpha}), u_{\mu,\alpha}) \leq C \|u_{\mu,\alpha}\|_2^2 \leq \Psi(u_{\mu,\alpha}) \\
 &\leq \Psi(c_{\mu,\alpha} d_{\mu,\alpha} w_\tau(\tau_{\mu,\alpha} s)) = C c_{\mu,\alpha}^2 d_{\mu,\alpha}^2 \tau_{\mu,\alpha}^{-N} \|w_\tau\|_2^2 \leq C \alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}} \mu_i^{-\frac{4}{N+2-p_i(N-2)}},
 \end{aligned}$$

this implies (3.6). Thus the proof is complete. \square

Lemma 3.3 Assume that $\{\mu, \alpha\} \subset R_+^{n+1}$ satisfies (C-i). Then $\lambda(\mu, \alpha)/\mu_i \rightarrow \infty$.

Proof. By Lemma 3.2 and (2.7)

$$(3.21) \quad \frac{\lambda(\mu, \alpha)}{\mu_i} \geq C (\alpha^2 \mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} \rightarrow \infty.$$

\square

Lemma 3.4 Assume that $\{\mu, \alpha\} \subset R_+^{n+1}$ satisfies (C-i). Then $\sigma_{\mu, \alpha} \rightarrow \infty$.

Proof. At first, we assume that there exists a subsequence of $\{\sigma_{\mu, \alpha}\}$ such that $\sigma_{\mu, \alpha} \rightarrow 0$ and derive a contradiction. We have by (2.8) that for $k \neq i$

$$(3.22) \quad \frac{\mu_k}{\mu_i} = o(1)(\alpha^2 \mu_i^{N-2})^{\frac{p_i - p_k}{N+2-p_i(N-2)}}.$$

Then we obtain by (1.6), (3.1) and Lemma 3.3 that

$$(3.23) \quad \begin{aligned} C(\alpha^2 \mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} &\leq \frac{\lambda(\mu, \alpha)}{\mu_i} < \sum_{k=1}^n \frac{\mu_k}{\mu_i} \frac{F_k(\sigma_{\mu, \alpha})}{G(\sigma_{\mu, \alpha})} \\ &\leq C(\sigma_{\mu, \alpha}^{q_i-1} + o(1) \sum_{k \neq i}^n (\alpha^2 \mu_i^{N-2})^{\frac{p_i - p_k}{N+2-p_i(N-2)}} \sigma_{\mu, \alpha}^{q_k-1}). \end{aligned}$$

This is a contradiction. Hence, $0 < \delta \leq \sigma_{\mu, \alpha}$. If there exists a subsequence of $\{\sigma_{\mu, \alpha}\}$ such that $\sigma_{\mu, \alpha} \leq C$, then by (3.21) and (3.22)

$$(3.24) \quad \begin{aligned} C(\alpha^2 \mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} &\leq \frac{\lambda(\mu, \alpha)}{\mu_i} < \sum_{k=1}^n \frac{\mu_k}{\mu_i} \frac{F_k(\sigma_{\mu, \alpha})}{G(\sigma_{\mu, \alpha})} \\ &\leq C(1 + o(1)(\alpha^2 \mu_i^{N-2})^{\frac{p_i - p_k}{N+2-p_i(N-2)}}). \end{aligned}$$

This contradicts (2.7). Thus the proof is complete. \square

Lemma 3.5 Assume that $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then for $1 \leq k \leq n$

$$(3.25) \quad \mu_k \sigma_{\mu, \alpha}^{p_k+1} \leq C \mu_i \sigma_{\mu, \alpha}^{p_i+1}.$$

Proof. We define $1 \leq j(\mu, \alpha) \leq n$ by the rule

$$(3.26) \quad \mu_{j(\mu, \alpha)} \sigma_{\mu, \alpha}^{p_{j(\mu, \alpha)}+1} = \max_{1 \leq k \leq n} \mu_k \sigma_{\mu, \alpha}^{p_k+1}.$$

Then there exists a infinite subsequence of $\{(\mu, \alpha)\}$ and $1 \leq j \leq n$ such that $j = j(\mu, \alpha)$ for all elements which belong to the subsequence. we consider this subsequence. We have by (3.1), (3.26) and Lemma 3.4 that for $0 \leq r \leq 1$

$$(3.27) \quad \begin{aligned} \frac{1}{2} u'_{\mu, \alpha}(r)^2 &\leq \sum_{k=1}^n \mu_k (F_k(\sigma_{\mu, \alpha}) - F_k(u_{\mu, \alpha}(r))) - \lambda(\mu, \alpha) (G(\sigma_{\mu, \alpha}) - G(u_{\mu, \alpha}(r))) \\ &\leq \sum_{k=1}^n \mu_k F_k(\sigma_{\mu, \alpha}) \leq C \sum_{k=1}^n \mu_k \sigma_{\mu, \alpha}^{p_k+1} \leq C \mu_j \sigma_{\mu, \alpha}^{p_j+1}. \end{aligned}$$

Let $0 \leq r_1 = r_{1, \mu, \alpha} \leq 1$ satisfy $u_{\mu, \alpha}(r_1) = 1/2 \sigma_{\mu, \alpha}$. By mean value theorem and (3.27)

$$\left| \frac{u_{\mu, \alpha}(0) - u_{\mu, \alpha}(r_1)}{r_1} \right| = \left| \frac{\sigma_{\mu, \alpha}}{2r_1} \right| \leq C \sqrt{\mu_j \sigma_{\mu, \alpha}^{p_j+1}},$$

that is,

$$(3.28) \quad C\sigma_{\mu,\alpha}^{\frac{1-p_j}{2}} \mu_j^{-1/2} \leq r_1.$$

Since $u_{\mu,\alpha}(r)$ is decreasing in $0 \leq r \leq 1$, it follows from (3.28) that

$$(3.29) \quad \begin{aligned} \|u_{\mu,\alpha}\|_2^2 &\geq \int_0^{r_1} r^{N-1} u_{\mu,\alpha}(r)^2 dr \geq C\sigma_{\mu,\alpha}^2 r_1^N \geq C\sigma_{\mu,\alpha}^2 \left(\sigma_{\mu,\alpha}^{\frac{1-p_j}{2}} \mu_j^{-1/2} \right)^N \\ &\geq C\sigma_{\mu,\alpha}^{\frac{4-N(p_j-1)}{2}} \mu_j^{-N/2}. \end{aligned}$$

This along with (3.20) implies that

$$(3.30) \quad \sigma_{\mu,\alpha} \leq C \left(\mu_j^{N/2} \alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}} \mu_i^{-\frac{4}{N+2-p_i(N-2)}} \right)^{\frac{2}{4-N(p_j-1)}}.$$

It follows from (1.5), (3.1) and Lemma 3.4 that

$$(3.31) \quad \lambda(\mu, \alpha) \leq C \sum_{k=1}^n \mu_k \sigma_{\mu,\alpha}^{p_k-1}.$$

Then by Lemma 3.3, (3.26) and (3.31)

$$\begin{aligned} C(\alpha^2 \mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} &\leq \frac{\lambda(\mu, \alpha)}{\mu_i} \leq C \mu_i^{-1} \mu_j \sigma_{\mu,\alpha}^{p_j-1} \\ &\leq C \mu_i^{-1} \mu_j \left(\mu_j^{N/2} \alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}} \mu_i^{-\frac{4}{N+2-p_i(N-2)}} \right)^{\frac{2(p_j-1)}{4-N(p_j-1)}} \\ &= C \mu_j^{\frac{4}{4-N(p_j-1)}} \alpha^{\frac{2(p_j-1)(4-N(p_i-1))}{(N+2-p_i(N-2))(4-N(p_j-1))}} \mu_i^{-\frac{8(p_j-1)}{(N+2-p_i(N-2))(4-N(p_j-1))}} \mu_i^{-1}, \end{aligned}$$

that is,

$$(3.32) \quad C \leq \mu_j \alpha^{\frac{2(p_j-p_i)}{N+2-p_i(N-2)}} \mu_i^{-\frac{N+2-p_j(N-2)}{N+2-p_i(N-2)}}.$$

Since we assume (2.8), we find that there never exists an infinite subsequence of $\{(\mu, \alpha)\}$ satisfying (3.26) for $j \neq i$, namely, (3.26) holds for $j = i$ except finite elements of $\{(\mu, \alpha)\}$. Thus we obtain our conclusion. \square .

Lemma 3.6 Assume that $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then

$$(3.33) \quad \lambda(\mu, \alpha) \leq C(\alpha \mu_i^{\frac{2}{p_i-1}})^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}.$$

Proof. By Lemma 3.4, Lemma 3.5 and (3.31)

$$(3.34) \quad \lambda(\mu, \alpha) \leq C \sum_{k=1}^n \mu_k \sigma_{\mu,\alpha}^{p_k-1} \leq C \mu_i \sigma_{\mu,\alpha}^{p_i-1}.$$

Since (3.30) holds for $j = i$, we have

$$(3.35) \quad \sigma_{\mu,\alpha} \leq C(\alpha^2 \mu_i^{N-2})^{1/(N+2-p_i(N-2))}.$$

Substituting (3.35) into (3.34), we obtain (3.33). \square

We conclude this section by showing the following lemma. Let $\xi_{\mu,\alpha} := (\lambda(\mu, \alpha)/\mu_i)^{1/(p_i-1)}$.

Lemma 3.7 *Assume that $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then*

$$(3.36) \quad C^{-1}\sigma_{\mu,\alpha} \leq \xi_{\mu,\alpha} \leq C\sigma_{\mu,\alpha}.$$

Proof. The second inequality follows from (3.34). By (3.21) and (3.35) we obtain

$$(3.37) \quad \sigma_{\mu,\alpha}^{p_i-1} \leq C(\alpha^2 \mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} \leq C \frac{\lambda(\mu, \alpha)}{\mu_i}.$$

Thus the proof is complete. \square

4 Proof of Theorem 2.1

We begin with recalling some fundamental properties of the ground state w of the equation (2.12) - (2.14). It is known that there uniquely exists the ground state w of (2.12)-(2.14) such that w is spherically symmetric, decreases with respect to $s = |x|$, $w \in C^2(R^N)$. Furthermore, for some constant $\delta > 0$ and for $|\gamma| \leq 2$

$$(4.1) \quad |D^\gamma w(x)| \leq C e^{-\delta|x|}, \quad x \in R^N$$

For these properties, we refer to Berestycki and Lions [1] and Kwong [6]. Since w is spherically symmetric, (2.12) - (2.14) are equivalent to:

$$(4.2) \quad w''(s) + \frac{N-1}{s} w'(s) + w(s)^{p_i} - w(s) = 0, \quad s > 0,$$

$$(4.3) \quad w(s) > 0, \quad s \geq 0,$$

$$(4.4) \quad \lim_{s \rightarrow \infty} w(s) = 0.$$

We put $w_{\mu,\alpha}(s) := \xi_{\mu,\alpha}^{-1} u_{\mu,\alpha}(r)$, $s := \sqrt{\lambda(\mu, \alpha)} r$. By (1.1) - (1.3) we see that $w_{\mu,\alpha}(s)$ satisfies the following equation:

$$(4.5) \quad \begin{aligned} & w''_{\mu,\alpha}(s) + \frac{N-1}{s} w'_{\mu,\alpha}(s) + w_{\mu,\alpha}(s)^{p_i} - w_{\mu,\alpha}(s) \\ & + \sum_{k=1, k \neq i}^n \lambda(\mu, \alpha)^{-1} \mu_k \xi_{\mu,\alpha}^{p_k-1} w_{\mu,\alpha}(s)^{p_k} \\ & + \sum_{k=1}^n \lambda(\mu, \alpha)^{-1} \mu_k \xi_{\mu,\alpha}^{-1} h_k(\xi_{\mu,\alpha} w_{\mu,\alpha}(s)) \end{aligned}$$

$$\begin{aligned}
(4.6) \quad & -\xi_{\mu,\alpha}^{-1} h_0(\xi_{\mu,\alpha} w_{\mu,\alpha}(s)) = 0, \quad s \in I_{\mu,\alpha} := (0, \sqrt{\lambda(\mu, \alpha)}), \\
& w_{\mu,\alpha}(s) > 0, \quad 0 \leq s < \sqrt{\lambda(\mu, \alpha)}, \\
(4.7) \quad & w'_{\mu,\alpha}(0) = 0, \quad w_{\mu,\alpha}(\sqrt{\lambda(\mu, \alpha)}) = 0.
\end{aligned}$$

By definition of $w_{\mu,\alpha}(s)$, we have:

$$(4.8) \quad \|w_{\mu,\alpha}\|_{X,\lambda}^2 := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w'_{\mu,\alpha}(s)^2 ds = \lambda(\mu, \alpha)^{\frac{N-2}{2}} \xi_{\mu,\alpha}^{-2} \|u_{\mu,\alpha}\|_X^2,$$

$$(4.9) \quad \|w_{\mu,\alpha}\|_{p_k+1,\lambda}^{p_k+1} := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w_{\mu,\alpha}(s)^{p_k+1} ds = \lambda(\mu, \alpha)^{\frac{N}{2}} \xi_{\mu,\alpha}^{-(p_k+1)} \|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1},$$

$$(4.10) \quad \|w_{\mu,\alpha}\|_{2,\lambda}^2 := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w_{\mu,\alpha}(s)^2 ds = \lambda(\mu, \alpha)^{\frac{N}{2}} \xi_{\mu,\alpha}^{-2} \|u_{\mu,\alpha}\|_2^2.$$

Furthermore, by (3.1), Lemma 3.5 and Lemma 3.7

$$\begin{aligned}
& \frac{1}{2} \lambda(\mu, \alpha) \xi_{\mu,\alpha}^2 w'_{\mu,\alpha}(s)^2 = \frac{1}{2} u'_{\mu,\alpha}(r)^2 \leq \sum_{k=1}^n \mu_k (F_k(\sigma_{\mu,\alpha}) - F_k(u_{\mu,\alpha}(r))) \\
& \leq C \sum_{k=1}^n \mu_k \sigma_{\mu,\alpha}^{p_k+1} \leq C \mu_i \sigma_{\mu,\alpha}^{p_i+1} \leq C \mu_i \xi_{\mu,\alpha}^{p_i+1} = C \lambda(\mu, \alpha) \xi_{\mu,\alpha}^2;
\end{aligned}$$

that is, for $s \in [0, \sqrt{\lambda(\mu, \alpha)}]$

$$(4.11) \quad w'_{\mu,\alpha}(s)^2 \leq C.$$

Lemma 4.1 Assume that $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then

$$(4.12) \quad \lambda(\mu, \alpha)^{-1} \mu_k \xi_{\mu,\alpha}^{p_k-1} w_{\mu,\alpha}(s)^{p_k} \rightarrow 0 \quad (k \neq i),$$

$$(4.13) \quad \lambda(\mu, \alpha)^{-1} \mu_k \xi_{\mu,\alpha}^{-1} h_k(\xi_{\mu,\alpha} w_{\mu,\alpha}(s)) \rightarrow 0 \quad (1 \leq k \leq n),$$

$$(4.14) \quad \xi_{\mu,\alpha}^{-1} h_0(\xi_{\mu,\alpha} w_{\mu,\alpha}(s)) \rightarrow 0.$$

for a fixed $s \in [0, \sqrt{\lambda(\mu, \alpha)}]$.

This lemma can be obtained by direct calculation. Hence, we omit the proof.

Lemma 4.2 Assume that $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then $\|w_{\mu,\alpha}\|_{2,\lambda}, \|w_{\mu,\alpha}\|_{X,\lambda} \leq C$.

Proof. By (1.4) and (2.4)

$$\begin{aligned}
(4.15) \quad \epsilon_1 \sum_{k=1}^n \mu_k \Phi_k(u_{\mu,\alpha}) & \leq 2\alpha + \epsilon_1 \sum_{k=1}^n \Phi_k(u_{\mu,\alpha}) \leq C \lambda(\mu, \alpha) (g(u_{\mu,\alpha}), u_{\mu,\alpha}) \\
& \leq C \lambda(\mu, \alpha) \|u_{\mu,\alpha}\|_2^2.
\end{aligned}$$

Since $u_{\mu,\alpha} \in N_{\mu,\alpha}$, we obtain by (3.20), (4.8) and (4.15) that

$$\begin{aligned}
(4.16) \quad \lambda(\mu, \alpha)^{\frac{2-N}{2}} \xi_{\mu,\alpha}^2 \|w_{\mu,\alpha}\|_{X,\lambda}^2 & = \|u_{\mu,\alpha}\|_X^2 = 2 \left(\sum_{k=1}^n \mu_k \Phi_k(u_{\mu,\alpha}) - \alpha \right) \leq 2 \sum_{k=1}^n \mu_k \Phi_k(u_{\mu,\alpha}) \\
& \leq 2C \epsilon_1^{-1} \lambda(\mu, \alpha) \|u_{\mu,\alpha}\|_2^2 \leq C \lambda(\mu, \alpha) \alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)} \mu_i^{-\frac{4}{N+2-p_i(N-2)}}};
\end{aligned}$$

this along with Lemma 3.2 implies that

$$\begin{aligned}
 (4.17) \quad \|w_{\mu,\alpha}\|_{X,\lambda}^2 &\leq C\lambda(\mu,\alpha)\xi_{\mu,\alpha}^{-2}\alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{-\frac{4}{N+2-p_i(N-2)}} \\
 &= C\left\{\lambda(\mu,\alpha)^{-1}\alpha^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{\frac{4}{N+2-p_i(N-2)}}\right\}^{\frac{4-N(p_i-1)}{2(p_i-1)}} \leq C.
 \end{aligned}$$

Next, by Lemma 3.2, (3.20) and (4.10)

$$\begin{aligned}
 \|w_{\mu,\alpha}\|_{2,\lambda}^2 &= \lambda(\mu,\alpha)^{N/2}\xi_{\mu,\alpha}^{-2}\|u_{\mu,\alpha}\|_2^2 \leq C\lambda(\mu,\alpha)^{N/2}\xi_{\mu,\alpha}^{-2}\alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{-\frac{4}{N+2-p_i(N-2)}} \\
 &= C\left\{\lambda(\mu,\alpha)^{-1}\alpha^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{\frac{4}{N+2-p_i(N-2)}}\right\}^{\frac{4-N(p_i-1)}{2(p_i-1)}} \leq C.
 \end{aligned}$$

Thus the proof is complete. \square

The following two lemmas are variant of Shibata [7, Lemma 4.7]. Hence, the proofs are omitted.

Lemma 4.3 *Let $w = w(|x|)$ be the ground state of (5.2) - (5.4). Suppose that $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then $\lim w_{\mu,\alpha}(|x|) = w(|x|)$ uniformly on any compact subsets on R^N .*

Lemma 4.4 *Assume that $\{(\mu, \alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then there exists $y_0(x) = y_0(|x|) \in L^2(R^N) \cap L^{p_k+1}(R^N)$ ($1 \leq k \leq n$) such that $w_{\mu,\alpha}(|x|) \leq y_0(|x|)$ for $x \in R^N$.*

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1: From Lemma 4.3, Lemma 4.4 and Lebesgue's convergence theorem

$$(4.18) \quad \|w_{\mu,\alpha}\|_{2,\lambda} \rightarrow \|w\|_{2,R^N}, \quad \|w_{\mu,\alpha}\|_{p_k+1,\lambda} \rightarrow \|w\|_{p_k+1,R^N} \quad (1 \leq k \leq n).$$

We know from Pohozaev's identity (cf. Strauss [8]) that

$$(4.19) \quad \|w\|_{p_i+1,R^N}^{p_i+1} = \frac{2(p_i+1)}{2+N-p_i(N-2)} \|w\|_{2,R^N}^2.$$

Furthermore, by Lemma 3.7, Lemma 4.2 and (4.10)

$$\begin{aligned}
 (4.20) \quad (g(u_{\mu,\alpha}), u_{\mu,\alpha}) &= \|u_{\mu,\alpha}\|_2^2 + \int_0^1 r^{N-1} h_0(u_{\mu,\alpha}(r)) u_{\mu,\alpha}(r) dr \\
 &= \lambda(\mu,\alpha)^{-N/2} \xi_{\mu,\alpha}^2 \|w_{\mu,\alpha}\|_{2,\lambda}^2 \\
 &\quad + \lambda(\mu,\alpha)^{-N/2} \xi_{\mu,\alpha}^2 \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} \frac{h_0(\xi_{\mu,\alpha} w_{\mu,\alpha}(s))}{\xi_{\mu,\alpha} w_{\mu,\alpha}(s)} w_{\mu,\alpha}(s)^2 ds \\
 &= (1 + o(1)) \lambda(\mu,\alpha)^{-N/2} \xi_{\mu,\alpha}^2 \|w_{\mu,\alpha}\|_{2,\lambda}^2.
 \end{aligned}$$

Similarly, we obtain

$$(4.21) \quad \begin{aligned} (f_k(u_{\mu,\alpha}), u_{\mu,\alpha}) &= \|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1} + \int_0^1 r^{N-1} h_k(u_{\mu,\alpha}(r)) u_{\mu,\alpha}(r) dr \\ &= (1 + o(1)) \lambda(\mu, \alpha)^{-N/2} \xi_{\mu,\alpha}^{p_k+1} \|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}, \end{aligned}$$

$$(4.22) \quad \Phi_k(u_{\mu,\alpha}) = \frac{1}{p_k + 1} (1 + o(1)) \lambda(\mu, \alpha)^{-N/2} \xi_{\mu,\alpha}^{p_k+1} \|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}.$$

By (2.4) and (4.20) - (4.22) we obtain

$$(4.23) \quad (1 + o(1)) \lambda(\mu, \alpha)^{(2-N)/2} \xi_{\mu,\alpha}^2 \|w_{\mu,\alpha}\|_2^2 = 2\alpha + \sum_{k=1}^n \frac{p_k - 1}{p_k + 1} (1 + o(1)) \mu_k \lambda(\mu, \alpha)^{-N/2} \xi_{\mu,\alpha}^{p_k+1} \|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1},$$

this implies that

$$(4.24) \quad \{A_1(\mu, \alpha) - A_2(\mu, \alpha)\} \lambda(\mu, \alpha)^{\frac{(N+2)-p_i(N-2)}{2(p_i-1)}} \mu_i^{-\frac{2}{p_i-1}} = 2\alpha,$$

where

$$(4.25) \quad A_1(\mu, \alpha) = (1 + o(1)) \|w_{\mu,\alpha}\|_{2,\lambda}^2,$$

$$(4.26) \quad \begin{aligned} A_2(\mu, \alpha) &= \frac{p_i - 1}{p_i + 1} (1 + o(1)) \|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1} \\ &\quad + \sum_{k \neq i}^n \frac{p_k - 1}{p_k + 1} (1 + o(1)) \mu_k \lambda(\mu, \alpha)^{-1} \xi_{\mu,\alpha}^{p_k-1} \|w_{\mu,\alpha}\|_{p_k+1,\lambda}^{p_k+1}. \end{aligned}$$

Then by Lemma 4.1, (4.18), (4.19) and (4.24) we obtain

$$(4.27) \quad \frac{\lambda(\mu, \alpha)}{\alpha^{\tau_1} \mu_i^{\tau_2}} \rightarrow \left(\frac{1}{\|w\|_{2,R^N}^2 - \frac{p_i-1}{p_i+1} \|w\|_{p_i+1,R^N}^{p_i+1}} \right)^{\tau_1} = \left(\frac{(N+2) - p_i(N-2)}{(4 + N - Np_i) \|w\|_{2,R^N}^2} \right)^{\tau_1},$$

where

$$\tau_1 = \frac{2(p_i - 1)}{N + 2 - (N - 2)p_i}, \quad \tau_2 = \frac{4}{N + 2 - (N - 2)p_i}.$$

Thus the proof is complete. \square

References

- [1] H. Berestycki, P. L. Lions, Nonlinear Scalar Field Equations, I Existence of a Ground State, Arch. Rational Mech. Analysis **82** (1983) 313–345.
- [2] R. Chiappinelli, Remarks on bifurcation for elliptic operators with odd nonlinearity, Israel J. Math. **65** (1989) 285–292.
- [3] M. Faierman, Two-parameter eigenvalue problems in ordinary differential equations, Longman House, Essex, UK (1991).

- [4] B. Gidas, W. M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979) 209–243.
- [5] D. Gilberg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, New York (1983).
- [6] M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^N , *Arch. Rational Mech. Analysis* **105** (1989) 243–266.
- [7] T. Shibata, Asymptotic behavior of eigenvalues of two-parameter nonlinear Sturm-Liouville problems, *J. Analyse Math.* **66** (1995) 277–294.
- [8] W. A. Strauss, Existence of solitary waves in higher dimensions, *Commun. Math. Phys.* **55** (1977) 149–162.
- [9] L. Turin, Sturm-Liouville problems with several parameters, *J. Differential Equations* **38** (1980) 239–259.
- [10] E. Zeidler, Ljusternik-Schnirelman theory on general level sets, *Math. Nachr.* **129** (1986) 235–259.